Appendix A

Laplace Transform Reference and Examples

A.1 Introduction

This document covers a basic introduction to forward and inverse Laplace Transforms. It will also present example problems using Laplace transforms to solve a mechanical system and an electrical system, respectively.

A.2 Synthesis and Analysis Equations

There are two main kinds of Laplace transform - the *bilateral* Laplace transform and the *unilateral* Laplace transform. The primary distinction between the two is that the unilateral Laplace transform only uses the portion of a signal after time 0^-. Anything else about the signal for negative times will be summarized by a constant in the transformation. Because of this, there are two different sets of synthesis and analysis equations:

	Bilateral	Unilateral
Synthesis	$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\ ds$	$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \mathcal{X}(s)e^{st}\ ds$
Analysis	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\ dt$	$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}\ dt$

Note in the synthesis equations that there is a constant real value σ in the limits on integration - this value is chosen such that the integral converges. The set of σ values for which the synthesis integral converges is known as the *Region of Convergence*, or *ROC*, of the particular transform. Also note that the synthesis equation itself is rarely used; rather, specific inverse Laplace transform pairs should be memorized and applied when appropriate.

Some books will clearly distinguish between the two kinds of Laplace transforms while others will simply assume one or the other (or use the same symbol for both!). The Oppenheim and Willsky book referenced below uses $X(s)$ to denote the bilateral Laplace transform and $\mathcal{X}(s)$ to denote the unilateral Laplace transform. The Haykin and Van Veen book uses $X(s)$ for both.

A.3 Common Laplace Transform Pairs and Properties

The next three subsections present tables of common Laplace transform pairs and Laplace transform properties. The information in these tables has been adapted from:

- **Signals and Systems**, 2nd ed. Simon Haykin and Barry Van Veen. John Wiley & Sons, Hoboken, NJ, 2005. pp. 781-783.
- **Signals and Systems**, 2nd ed. Alan V. Oppenheim and Alan S. Willsky with S. Hamid Nawab. Prentice Hall, Upper Saddle River, NJ, 1997. p. 691-692.

A.3.1 Common Laplace Transform Pairs

Basic Bilateral Laplace Transform Pairs

Name	**Signal**	**Laplace Transform**	**ROC**
Basic Signal	$x(t)$	$X(s)$	R_x
Impulse	$x(t) = \delta(t - t_0)$	$X(s) = e^{-st_0}$	All s
Unit step	$x(t) = u(t - t_0)$	$X(s) = \dfrac{e^{-st_0}}{s}$	$\sigma > 0$
Reversed step	$x(t) = -u(-(t - t_0))$	$X(s) = \dfrac{e^{-st_0}}{s}$	$\sigma < 0$
Polynomial	$x(t) = \dfrac{t^{n-1}}{(n-1)!}u(t)$	$X(s) = \dfrac{1}{s^n}$	$\sigma > 0$
Reversed Polynomial	$x(t) = -\dfrac{t^{n-1}}{(n-1)!}u(-t)$	$X(s) = \dfrac{1}{s^n}$	$\sigma < 0$
Exponential	$x(t) = e^{-\alpha t}u(t)$	$X(s) = \dfrac{1}{s+\alpha}$	$\sigma > -\Re\{\alpha\}$
Reversed Exponential	$x(t) = -e^{-\alpha t}u(-t)$	$X(s) = \dfrac{1}{s+\alpha}$	$\sigma < -\Re\{\alpha\}$
Polynomial Exponential	$x(t) = \dfrac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$X(s) = \dfrac{1}{(s+\alpha)^n}$	$\sigma > -\Re\{\alpha\}$
Rev. Poly. Exp.	$x(t) = -\dfrac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$X(s) = \dfrac{1}{(s+\alpha)^n}$	$\sigma < -\Re\{\alpha\}$
Cosine	$x(t) = \cos(\omega_0 t)u(t)$	$X(s) = \dfrac{s}{s^2+\omega_0^2}$	$\sigma > 0$
Sine	$x(t) = \sin(\omega_0 t)u(t)$	$X(s) = \dfrac{\omega_0}{s^2+\omega_0^2}$	$\sigma > 0$
Exponential Cosine	$x(t) = e^{-\alpha t}\cos(\omega_0 t)u(t)$	$X(s) = \dfrac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\sigma > -\Re\{\alpha\}$
Exponential Sine	$x(t) = e^{-\alpha t}\sin(\omega_0 t)u(t)$	$X(s) = \dfrac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\sigma > -\Re\{\alpha\}$

A.3.2 Common Laplace Transform Properties

For the most part, the unilateral Laplace transform properties are the same as those for the bilateral Laplace transform. The major exceptions have to do with the fact that the unilateral Laplace transform is only defined for the part of the signal that exists at or after $t = 0^-$, so any property that could shift, scale, or reverse the signal will have a slightly different form. Also, the derivative property requires knowledge of the initial value since there will be an instantaneous change as a result. Finally, the integral property has a lower limit of 0^- instead of $-\infty$. The bilateral properties are presented below and the unilateral properties are on the next page.

Properties of the Bilateral Laplace Transform

Property	**Signal**	**Laplace Transform**	**ROC**
Basic Signals	$x(t), y(t), z(t)$	$X(s), Y(s), Z(s)$	R_x, R_y, R_z
Linearity	$z(t) = Ax(t) + By(t)$	$Z(s) = AX(s) + BY(s)$	At least $R_x \cap R_y$
Time Shifting	$z(t) = x\left(t - t_0\right)$	$Z(s) = e^{-st_0}X(s)$	R_x
s-domain Shifting	$z(t) = e^{s_0 t}x(t)$	$Z(s) = X(s - s_0)$	s for $s - s_0 \in R_x$
Conjugation	$z(t) = x^*(t)$	$Z(s) = X^*(s^*)$	R_x
Time and Frequency Scaling	$z(t) = x(at)$	$Z(s) = \frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	s for $\frac{s}{a} \in R_x$
Convolution	$z(t) = x(t) * y(t)$	$Z(s) = X(s)Y(s)$	At least $R_x \cap R_y$
Time Differentiation	$z(t) = \frac{d}{dt}x(t)$	$Z(s) = sX(s)$	At least R_x
Integration	$z(t) = \int_{-\infty}^{t} x(\tau)d\tau$	$Z(s) = \frac{1}{s}X(s)$	At least $R_x \cap \{\sigma > 0\}$
Frequency Differentiation	$z(t) = -tx(t)$	$Z(s) = \frac{d}{ds}X(s)$	R_x

Properties of the Unilateral Laplace Transform

Property	**Signal**	**Laplace Transform**
Basic Signals	$x(t), y(t), z(t)$ $x(t) = y(t) = 0,\ t < 0$	$\mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s)$
Linearity	$z(t) = Ax(t) + By(t)$	$\mathcal{Z}(s) = A\mathcal{X}(s) + B\mathcal{Y}(s)$
Time Shifting	$z(t) = x\left(t - t_0\right)$	$\mathcal{Z}(s) = e^{-st_0}\mathcal{X}(s)$ if $x(t-t_0)u(t) = x(t-t_0)u(t-t_0)$
s-domain Shifting	$z(t) = e^{s_0 t}x(t)$	$\mathcal{Z}(s) = \mathcal{X}(s - s_0)$
Time and Frequency Scaling	$z(t) = x(at),\ a > 0$	$\mathcal{Z}(s) = \frac{1}{a}\mathcal{X}\left(\frac{s}{a}\right)$
Conjugation	$z(t) = x^*(t)$	$\mathcal{Z}(s) = \mathcal{X}^*(s^*)$
Convolution	$z(t) = x(t) * y(t)$	$\mathcal{Z}(s) = \mathcal{X}(s)\mathcal{Y}(s)$
Time Differentiation	$z(t) = \frac{d}{dt}x(t)$	$Z(s) = s\mathcal{X}(s) - x(0^-)$
Frequency Differentiation	$z(t) = -tx(t)$	$\mathcal{Z}(s) = \frac{d}{ds}\mathcal{X}(s)$
Time Integration	$z(t) = \int_{0^-}^{t} x(\tau)d\tau$	$\mathcal{Z}(s) = \frac{1}{s}\mathcal{X}(s)$

A.3.3 Laplace Transform Initial and Final Value Theorems

The initial and final value for a signal $x(t)$ can be determined by its Laplace transform $X(s)$ if certain conditions are met. If $X(s)$ is defined as a ratio of polynomials, and the highest order of the numerator is less than the highest order of the denominator, then the initial value for the signal can be determined with:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Note that if the numerator is of the same or higher order than the denominator of the Laplace transform the limit on the right does not converge.

If $X(s)$ is defined as a ratio of polynomials, and all the values of s that make the denominator equal to 0 have negative real parts (in other words, all the poles are in the left half-plane), then the final value for the signal can be determined with:

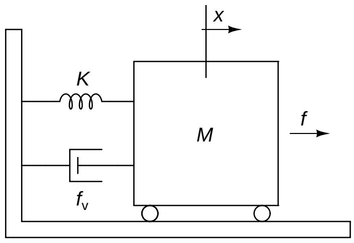
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

A.4 Examples

The following are examples of how Fourier series coefficients can be used to simplify and solve for outputs (in these cases, positions and output voltages) given periodic inputs (forces and input voltages).

A.4.1 Spring-Mass-Damper System

A mass M is attached to a stationary wall via a spring K and a damper f_v. An external force $f(t)$ is applied to the mass and the resulting position of the mass $x(t)$ is measured.



The external force is given by the equation:

$$f(t) = \left(\alpha + \beta\cos(5t) + \gamma e^{-3t}\right)\ u(t)$$

where α, β, and γ are real constants. At time 0, the mass is displaced $x_0 = 1$ m and has a velocity of $\dot{x}_0 = 3$ m/s. The element properties are: M=1 kg, f_v=1010 kg/s=1010 N·s/m and K=10000 kg/s^2=10000 N/m.

Use the bilateral Laplace transform to find the transfer function between the output position and the input force, then use the unilateral Laplace transform to determine a function for the output position.

Determine the Differential Equation and Transfer Function

For this mass, the equation of motion is given by:

$$\sum F_x = -Kx - f_v\frac{dx}{dt} + f(t) = M\frac{d^2x}{dt^2}$$
$$M\frac{d^2x}{dt^2} + Kx + f_v\frac{dx}{dt} = f(t)$$

Assuming the Laplace transform of the input force is given by $F(s)$ and the Laplace transform of the output position is given by $X(s)$, use the differentiation property to obtain the transfer function $H(s) = X(s)/F(s)$:

$$(Ms^2 + f_v s + K)X(s) = F(s)$$
$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

The transfer function can be used to determine how the system will respond to a wide variety of inputs, but as it is based on the bilateral transform, any solution assumes that $x(t)$ and all its derivatives at time 0^- are zero (or, alternately, that you are looking only for the particular solution). To get the complete solution, you need to use the unilateral transform properties with the differential equation.

Determine the Unilateral Laplace Transform of the Output

Using the differential equation:

$$M\frac{d^2x}{dt^2} + Kx + f_v\frac{dx}{dt} = f(t)$$

the unilateral transform properties produce the relationship:

$$M(s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-)) + f_v((s\mathcal{X}(s) - x(0^-)) + K(\mathcal{X}(s)) = \mathcal{F}(s)$$
$$(Ms^2 + f_v s + K)\mathcal{X}(s) = Msx(0^-) + M\dot{x}(0^-) + f_v x(0^-) + \mathcal{F}(s)$$
$$\mathcal{X}(s) = \frac{Msx(0^-) + M\dot{x}(0^-) + f_v x(0^-) + \mathcal{F}(s)}{Ms^2 + f_v s + K}$$

Note that this is *very* similar to what the transfer function would produce, with the exception of the initial position and velocity terms in the numerator. Those represent the homogeneous portion of the complete solution. Now substitute in the Laplace transform of the input:

$$\begin{aligned}
\mathcal{F}(s) &= \mathcal{UL}\left\{f(t)\right\} \\
\mathcal{F}(s) &= \mathcal{UL}\left\{\left(\alpha + \beta\cos(5t) + \gamma e^{-3t}\right)\ u(t)\right\} \\
\mathcal{F}(s) &= \frac{\alpha}{s} + \frac{\beta s}{s^2+5^2} + \frac{\gamma}{s+3} \\
\mathcal{F}(s) &= \frac{\alpha\, s^3 + 3\,\alpha\, s^2 + 25\,\alpha\, s + 75\,\alpha + \beta\, s^3 + 3\,\beta\, s^2 + \gamma\, s^3 + 25\,\gamma\, s}{s\,(s^2+25)\,(s+3)}
\end{aligned}$$

which, once put into the expression for $\mathcal{X}(s)$ above *and simplified*, gives:

$$\mathcal{X}(s) = \frac{Ms^5x_0 + (M\dot{x}_0 + f_vx_0 + 3\,Mx_0)\,s^4 + (\gamma + \alpha + 3\,f_vx_0 + \beta + 3\,M\dot{x}_0 + 25\,Mx_0)\,s^3}{s\,(s^2+25)\,(s+3)\,(Ms^2+f_vs+K)} + \frac{(3\,\alpha + 3\,\beta + 25\,f_vx_0 + 75\,Mx_0 + 25\,M\dot{x}_0)\,s^2 + (25\,\alpha + 75\,M\dot{x}_0 + 25\,\gamma + 75\,f_vx_0)\,s + 75\,\alpha}{s\,(s^2+25)\,(s+3)\,(Ms^2+f_vs+K)}$$

substituting in for the parameter values and initial conditions gives:

$$\mathcal{X}(s) = \frac{s^5 + 1016\,s^4 + (\gamma + \alpha + 3064 + \beta)\,s^3 + (3\,\alpha + 3\,\beta + 25400)\,s^2 + (25\,\alpha + 75975 + 25\,\gamma)\,s + 75\,\alpha}{s\,(s^2+25)\,(s+3)\,(s^2+1010\,s+10000)}$$

Using partial fraction expansion on this expression yields:

$$\begin{aligned}
\mathcal{X}(s) = \cdots& \\
&\frac{0.0001\,\alpha}{s} + \\
&\frac{0.0000143\,\gamma}{s+3.0} + \\
&\frac{1.01 - 0.0000808\,\beta - 0.000144\,\gamma - 0.000101\,\alpha}{s+10.0} + \\
&\frac{-0.0131 + 0.00000101\,\beta + 0.00000101\,\gamma + 0.00000101\,\alpha}{s+1000.0} + \\
&\frac{0.0000798\,\beta\,s + 0.000202\,\beta}{s^2+25.0}
\end{aligned}$$

This form is useful because the denominators will indicate the *form* - though not the magnitude - of the functions that will comprise the solution. The numerators will give the relative magnitudes - and in the case of the last part, the balance between cosine and sine. The five denominators, in order, are indicative of a constant, an exponential with a decay rate of 3, an exponential with a decay rate of 10, an exponential with a decay rate of 1000, and sine and cosine terms with a frequency of 5 rad/s.

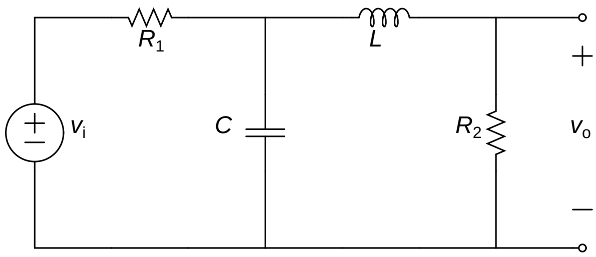
The final answer, obtained by taking the inverse Laplace transform of the expression above, is:

$$\begin{aligned}
x(t) = \cdots& \\
&0.0001\,\alpha + \\
&0.000143\,\gamma\, e^{-3.0\,t} + \\
&(1.01 - 0.0000808\,\beta - 0.000144\,\gamma - 0.000101\,\alpha)\, e^{-10.0\,t} + \\
&(-0.0131 + 0.00000101\,\beta + 0.00000101\,\gamma + 0.00000101\,\alpha)\, e^{-1000.0\,t} + \\
&0.0000798\,\beta\cos(5.0\,t) + 0.0000404\,\beta\sin(5.0\,t)
\end{aligned}$$

when $t \geq 0$.

A.4.2 RLC Circuit

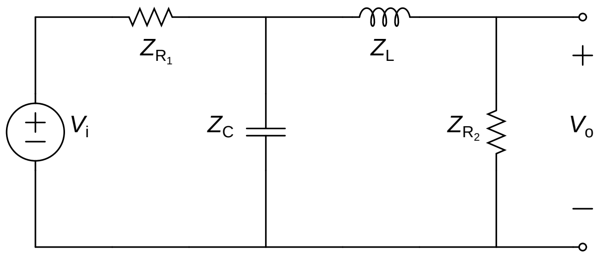
The example above, while demonstrating the full formal process for using Laplace transforms, gets rapidly bogged down with algebra. Typically, Laplace transforms for mechanical and electrical systems with multiple degrees of freedom will do that, so it is a good idea to use Laplace transforms in conjunction with a program capable of symbolically solving simultaneous equations as well as being able to solve forward and inverse Laplace transforms. Furthermore, to solve systems with initial conditions other than 0, you will either need to carry around the extra terms from the unilateral Laplace transform *or* solve the problem in frequency space using the bilateral transform, use the differentiation property to reconstruct a differential equation, and determine enough initial conditions to properly solve the differential equation. The following discussion will cover the latter case using the circuit below:



with element values $R_1 = R_2$=2 kΩ, L=1 mH, C=2.2 μF.

Convert to Impedances and Laplace Transforms

Converting the inductor, capacitor, and resistances to their impedances and re-writing the variables in terms of their Laplace transforms yields the following circuit:



Set Up and Solve Circuit Equations

The circuit above can be solved with voltage division. If we define

$$\mathbb{Z}_{LR_2C} = \mathbb{Z}_C \| (\mathbb{Z}_L + \mathbb{Z}_{R_2})$$

then

$$V_o(s) = V_i(s) \cdot \left(\frac{\mathbb{Z}_{LR_2C}}{\mathbb{Z}_{R_1} + \mathbb{Z}_{LR_2C}} \right) \cdot \left(\frac{\mathbb{Z}_{R_2}}{\mathbb{Z}_L + \mathbb{Z}_{R_2}} \right)$$

Substituting in the individual impedances:

$$\mathbb{Z}_L = sL \qquad \mathbb{Z}_C = \frac{1}{sC}$$
$$\mathbb{Z}_{R_1} = R_1 \qquad \mathbb{Z}_{R_2} = R_2$$

and simplifying yields:

$$V_o(s) = V_i(s) \cdot \left(\frac{R_2}{(R_1\, L\, C)\, s^2 + (R_1\, R_2\, C + L)\, s + (R_1 + R_2)} \right)$$

Cross-multiplying produces:

$$\left((R_1\, L\, C)\, s^2 + (R_1\, R_2\, C + L)\, s + (R_1 + R_2)\right)\, V_o(s) = (R_2)\, V_i(s)$$

Convert to Differential Equation

The s terms in the coefficients represent derivatives, so the Laplace transform above represents the differential equations:

$$(R_1\,L\,C)\,\frac{d^2v_{\rm o}}{dt^2}+(R_1\,R_2\,C+L)\,\frac{dv_{\rm o}}{dt}+(R_1+R_2)\,v_{\rm o}=R_2v_{\rm i}$$

Since this is a second order ordinary differential equation, we will need to determine two values for $v_{\rm o}$ - typically the initial value and the initial derivative, though that is not explicitly required.

Determine Required Initial Conditions

The problem is, for circuits, the initial conditions are generally given in terms of capacitor voltages and inductor currents, since those are the variables that do not change instantaneously. In this case, for example, assume that the input voltage (in...volts) is given by:

$$v_{\rm i}(t)=\left\{\begin{array}{ll} t<0 & 3\\ t\geq 0 & 5+5\cos(10^3t)\end{array}\right.$$

and further assume that the system has been running for a very, very long time before time $t=0$. The initial capacitor voltage and inductor current can be determined using the DC equivalent circuit; assuming the capacitor voltage is measured from the top node to the bottom node and the inductor current is measured as flowing from left to right, they are:

$$v_{\rm C}(0^-)=v_{\rm i}(0^-)\left(\frac{R_2}{R_1+R_2}\right)=1.5\text{ V}\qquad\qquad i_{\rm L}(0^-)=\frac{v_{\rm i}(0^-)}{R_1+R_2}=0.75\text{ mA}$$

The values for $v_{\rm o}(0^+)$ and $\dot{v}_{\rm o}(0^+)$ will come from examining the circuit at time 0^+ and keeping in mind the fact that:

$$\begin{aligned} v_{\rm C}(0^+)&=v_{\rm C}(0^-)=1.5\text{ V}\\ i_{\rm L}(0^+)&=i_{\rm L}(0^-)=0.75\text{ mA}\end{aligned}$$

For this circuit, since the current through the inductor is known, the voltage across resistor R_2 - which is the same as the output voltage - can be determined using Ohm's Law:

$$v_{\rm o}(0^+)=v_{{\rm R}_2}(0^+)=i_{\rm L}(0^+)\,R_2=1.5\text{ V}$$

The more difficult equation here is to get $\dot{v}_{\rm o}(0^+)$. Note that this would also be the same as $\dot{i}_{\rm o}(0^+)\,R_2$. Since the resistor is in series with the inductor, $\dot{i}_{\rm o}(0^+)=\dot{i}_{\rm L}(0^+)=\frac{1}{L}v_{\rm L}(0^+)$. The voltage across an inductor, however, *can* change instantaneously so this does not give a known initial condition.

If you look at the circuit, the "easily" known values at time 0^+ are the source voltage, the capacitor voltage, the inductor current, and - because of the known inductor current - the voltage and current for R_2. Using KVL on the right loop yields:

$$\begin{aligned} -v_{\rm C}(0^+)+v_{\rm L}(0^+)+v_{{\rm R}_2}(0^+)&=0\\ v_{\rm L}(0^+)&=v_{\rm C}(0^+)-v_{{\rm R}_2}(0^+)=0\text{ V}\end{aligned}$$

For this circuit, the initial *voltage* across the inductor happens to be 0 V, meaning the derivative of its current is also initially zero. This in turn means the derivative of the voltage across resistor R_2 is similarly zero.

Solve Differential Equation

At this point, everything necessary for solving the problem at hand is known - there is a differential equation:

$$(R_1\,L\,C)\,\frac{d^2v_{\rm o}}{dt^2} + (R_1\,R_2\,C + L)\,\frac{dv_{\rm o}}{dt} + (R_1 + R_2)\,v_{\rm o} = R_2 v_{\rm i}$$

with element values $R_1 = R_2$=2 kΩ, L=1 mH, C=2.2 μF, a forcing function **for times greater than 0** of

$$v_{\rm i}(t) = 5 + 5\cos(10^3 t)$$

with initial conditions

$$v_{\rm o}(0^+) = 1.5 \text{ V}$$
$$\dot{v}_{\rm o}(0^+) = 0 \text{ V/s}$$

Substituting in for the parameter values and the function of the input gives:

$$0.0000044\,\frac{d^2v_{\rm o}}{dt^2} + 8.801\,\frac{dv_{\rm o}}{dt} + 4000.0\,v_{\rm o} = 2000.0\,\left(5 + 5\cos(10^3 t)\right)$$

Solving this yields

$$v_{\rm o}(t) = 2.5 - 1.4285e^{-454.56t} + 0.00079\,e^{-1.9998*10^6\,t} + 0.42769\,\cos\left(10^3\,t\right) + 0.94206\,\sin\left(10^3\,t\right)$$

for times $t \geq 0$.